

# ON FINITE TYPE AND FLAT EPIMORPHISMS OF RINGS

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**ABSTRACT.** This article is two folded but its topics are closely related to each other. In the first part, finite type epimorphisms of rings are completely characterized. In the second part, by using Gabriel localization theory, new progresses in the understanding the structure of flat epimorphisms of rings has been made. As a by-product of this study, a remarkable result in commutative algebra come to light.

## 1. INTRODUCTION

The main topics of the present article are “finite type epimorphisms” and “flat epimorphisms” of rings. As a first main result, in Theorem 3.4, finite type epimorphisms of rings are completely characterized. This algebraic result illuminates some geometric aspects of the finite type monomorphisms of schemes. In the second part of the article (§4), by applying the whole strength of [10], new progresses in the understanding the structure of flat epimorphisms has been made. This study reveals some new aspects of the epimorphisms to us which were not discovered in the Séminaire Samuel. We refer to [9], specially [4], [6] and [7] for a comprehensive discussion of the epimorphisms of commutative rings.

Daniel Lazard in [4, Proposition 3.4 ], associates to each ring  $R$  a maximal ring  $M(R)$  and a canonical ring map  $R \rightarrow M(R)$ . Maximal in the sense that every injective flat epimorphism with source  $R$  can be canonically embedded in  $M(R)$ . But there is a serious set theoretical gap in his construction. Indeed, in his method of the constructing  $M(R)$ , we need to know that the underlying set of the ring should be actually a “set”. But there is no proof and even no reference for this non-trivial fact at there. Hence we decided to inform Lazard about the existence of such a major problem in the body of the ring  $M(R)$

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in order to resolve it in a way. But at that time, Lazard did not have any idea that how to resolve this gap. After some effort we eventually succeeded to overcome the problem by applying another tool. In fact, more recently in [10, Corollary 3.3] it was shown that every flat epimorphism of rings could be realized by a Gabriel localization. Then, in Theorem 4.1, the foregoing set theoretical gap completely filled by using this realization. After removing this gap, then an explicit description of the ring  $M(R)$  was presented in Theorem 4.3. We call  $M(R)$  the *Lazard* ring of  $R$ . Gabriel localization theory not only resolved this problem but also by using this theory then we made some new progresses in the understanding the structure of the ring  $M(R)$ , see Theorems 4.4, 4.6 and Corollary 4.14. The important outcome is that the ring  $M(R)$  can be canonically embedded in the total ring of fractions of the polynomial ring  $R[x]$ . This result strongly bounds the size of the ring  $M(R)$  while it was previously believed that the ring  $M(R)$  should be significantly large because every  $R$ -algebra whose structure morphism is an injective flat epimorphism is living, as a subring, inside the ring  $M(R)$ . As a by-product of this study, a remarkable result in commutative algebra also come to light, see Theorem 4.11. We should mention that a very special case of Theorem 4.11 was announced in [8, Corollary 3.4.7]. The method of proving Theorem 4.11 is very similar to [2, Theorem 1.1]. But the proof which presented at [2] heavily based on other major results in the literature while our proof is just using the results of the present article.

After finishing, we sent a primary version of the present article together with [10] by an e-mail to Lazard and in that letter we also explained to him that now the foregoing gap has been completely resolved. Then after about two months we received a message from him. His letter in addition to that was including many valuable comments about the manuscripts, in partial of the letter, he had also evaluated these articles as follows: “in summary you have done great progresses in the understanding the structure of flat epimorphisms of rings”. Hearing such words from Lazard who is a specialist in commutative algebra was so pleasant to us.

In the second section of this article, we study the general properties of epimorphisms. All of the results of this section more or less are well-known. The titles of the final sections should be sufficiently explanatory. Indeed, the main results of these sections was described in the above paragraphs. Throughout the article, all of the rings which

are discussed are commutative.

## 2. PRELIMINARIES-EPICS CHARACTERIZATION

In this article, by an epimorphism  $\varphi : R \rightarrow S$  we mean it is an epimorphism in the category of commutative rings. Although the class of injective ring maps is precisely coincide to the class of monomorphisms of rings. But the surjective ring maps are just special cases of the epimorphisms. As a specific example, the canonical ring map  $\mathbb{Z} \rightarrow \mathbb{Q}$  is an epimorphism while it is not surjective. Many nice properties are equivalent to the concept of “epimorphism”:

**Theorem 2.1.** *Let  $\varphi : R \rightarrow S$  be a ring map. Then the following conditions are equivalent.*

- (i)  $\varphi$  is an epimorphism.
- (ii) In the ring  $S \otimes_R S$ ,  $s \otimes 1 = 1 \otimes s$  for all  $s \in S$ .
- (iii) The canonical ring map  $p : S \otimes_R S \rightarrow S$  which maps each pure tensor  $s \otimes s'$  into  $ss'$  is bijective.
- (iv) The ring map  $i : S \rightarrow S \otimes_R S$  defined by  $s \rightsquigarrow s \otimes 1$  is bijective.
- (v) We have  $S \otimes_R \text{Coker}(\varphi) = 0$ .
- (vi) The tensor algebra  $T_R(S) = \bigoplus_{p \geq 0} T^p(S)$  is a commutative ring.
- (vii) For each  $R$ -algebra  $T$  there is at most one homomorphism of  $R$ -modules  $f : S \rightarrow T$  such that  $f(1_S) = 1_T$ .
- (viii) The restriction of scalars functor  $\varphi_* : \mathbf{S}\text{-mod} \rightarrow \mathbf{R}\text{-mod}$  is full and faithful.
- (ix) For each  $S$ -module  $M$  the canonical map  $S \otimes_R M \rightarrow M$  given by  $s \otimes m \rightsquigarrow sm$  is bijective.

**Proof.** (i)  $\Rightarrow$  (ii) : We have  $i \circ \varphi = j \circ \varphi$  where  $i, j : S \rightarrow S \otimes_R S$  are the canonical ring maps which map each  $s \in S$  into  $s \otimes 1$  and  $1 \otimes s$ , respectively.

(ii)  $\Leftrightarrow$  (iii) : We have  $\text{Ker}(p) = \langle s \otimes 1 - 1 \otimes s : s \in S \rangle$ . Because if  $\sum_i s_i s'_i = 0$  then we may write  $\sum_i s_i \otimes s'_i = \sum_i 1 \otimes s'_i (s_i \otimes 1 - 1 \otimes s_i)$ .

(ii)  $\Rightarrow$  (iv) : The map  $i$  is always injective. Each pure tensor  $s \otimes s'$  can be written as  $s \otimes s' = (s \otimes 1)(1 \otimes s') = (s \otimes 1)(s' \otimes 1) = ss' \otimes 1 = i(ss')$ .

(iv)  $\Rightarrow$  (iii) : We have  $p \circ i = \text{Id}$ . Therefore  $p$  is bijective.

(iii)  $\Rightarrow$  (i) : Let  $f, g : S \rightarrow T$  be ring maps such that  $f \circ \varphi = g \circ \varphi$ . By the universal property of the pushouts, there is a (unique) ring map  $\psi : S \otimes_R S \rightarrow T$  such that  $f = \psi \circ i$  and  $g = \psi \circ j$ . But  $i = j$  since

$\text{Ker}(p) = 0$ . Thus  $f = g$ .

(iv)  $\Leftrightarrow$  (v) : Apply the right exact functor  $S \otimes_R -$  to the exact sequence  $R \xrightarrow{\varphi} S \xrightarrow{\pi} \text{Coker } \varphi \longrightarrow 0$  we then obtain the following

exact sequence  $S \otimes_R R \xrightarrow{1_S \otimes \varphi} S \otimes_R S \xrightarrow{1_S \otimes \pi} S \otimes_R \text{Coker } \varphi \longrightarrow 0$ . But

$\text{Im}(1_S \otimes \varphi) = \text{Im}(i)$ . Therefore  $i$  is onto if and only if  $S \otimes_R \text{Coker } \varphi = 0$ .

(ii)  $\Leftrightarrow$  (vi) : First assume that  $s \otimes 1 = 1 \otimes s$  for all  $s \in S$ . For any two pure tensors  $s_1 \otimes \dots \otimes s_p$  and  $s'_1 \otimes \dots \otimes s'_q$  in  $T_R(S)$  then its multiplication is defined as  $(s_1 \otimes \dots \otimes s_p) \cdot (s'_1 \otimes \dots \otimes s'_q) = s_1 \otimes \dots \otimes s_p \otimes s'_1 \otimes \dots \otimes s'_q$ . For each  $p \geq 1$  and for each pure tensor  $s_1 \otimes \dots \otimes s_p$ , we easily observe that  $s \cdot (s_1 \otimes \dots \otimes s_p) = (s_1 \otimes \dots \otimes s_p) \cdot s$ . Thus  $(s_1 \otimes \dots \otimes s_p) \cdot (s'_1 \otimes \dots \otimes s'_q) = (s'_1 \otimes \dots \otimes s'_q) \cdot (s_1 \otimes \dots \otimes s_p)$ . Therefore  $T_R(S)$  is a commutative ring. Conversely, in  $T_R(S)$  we have  $s \cdot s' = s' \cdot s$ . Therefore  $s \otimes s' = s' \otimes s$ . In particular, for  $s' = 1$ ,  $s \otimes 1 = 1 \otimes s$ .

(ii)  $\Rightarrow$  (vii) : Let  $f, g : S \rightarrow T$  be two  $R$ -homomorphisms such that  $f(1_S) = g(1_S) = 1_T$ . The  $R$ -bilinear map  $S \times S \rightarrow T$  given by  $(s, s') \rightsquigarrow f(s)g(s')$ , by the universal property of the tensor products, induces a (unique) homomorphism of  $R$ -modules  $S \otimes_R S \rightarrow T$  such that  $s \otimes s'$  is mapped into  $f(s)g(s')$ . But  $s \otimes 1 = 1 \otimes s$  and so  $f(s) = g(s)$  for all  $s \in S$ .

(vii)  $\Rightarrow$  (i) : Every homomorphism of  $R$ -algebras is a homomorphism of  $R$ -modules.

(ii)  $\Rightarrow$  (viii) : The functor  $\varphi_*$  is always faithful. Let  $M$  and  $N$  be  $S$ -modules and let  $f : M \rightarrow N$  be a  $R$ -linear map. We show that it is also  $S$ -linear. For each  $m \in M$ , the  $R$ -bilinear map  $S \times S \rightarrow N$  given by  $(s, s') \rightsquigarrow sf(s'm)$ , by the universal property of the tensor products, induces a (unique)  $R$ -homomorphism  $S \otimes_R S \rightarrow N$  which maps each pure tensor  $s \otimes s'$  into  $sf(s'm)$ . But  $1 \otimes s = s \otimes 1$  and so  $f(sm) = sf(m)$  which was desired.

(viii)  $\Rightarrow$  (ii) : We consider  $S \otimes_R S$  as  $S$ -module where the scalar multiplication is defined on pure tensors as  $s \cdot (s' \otimes s'') = s' \otimes ss''$ . By the hypothesis, the canonical map  $i : S \rightarrow S \otimes_R S$  is  $S$ -linear. This, in particular, implies that  $s \otimes 1 = 1 \otimes s$  for all  $s \in S$ .

(ii)  $\Rightarrow$  (ix) : Recall that for any two  $S$ -modules  $M$  and  $N$ ,  $M \otimes_R N$  naturally has a  $S \otimes_R S$ -module structure where the scalar multiplication is defined on pure tensors as  $(s \otimes s') \cdot (m \otimes n) = sm \otimes s'n$ . If  $sm = 0$  then, by the hypothesis, we may write  $s \otimes m = (s \otimes 1) \cdot (1 \otimes m) = (1 \otimes s) \cdot (1 \otimes m) = 1 \otimes sm = 0$ .

(ix)  $\Rightarrow$  (iii) : There is nothing to prove.  $\square$

The above Theorem has the following important consequences.

**Corollary 2.2.** *Every faithfully flat epimorphism of rings is an isomorphism.*

**Proof.** Every faithfully flat ring map is injective since for any ring map  $\varphi : R \rightarrow S$  the map  $1_S \otimes \varphi : S \otimes_R R \rightarrow S \otimes_R S$  is injective. If  $\varphi$  is an epimorphism then by Theorem 2.1,  $S \otimes_R \text{Coker } \varphi = 0$ . This implies that  $\text{Coker } \varphi = 0$  since  $S$  is faithfully flat over  $R$ .  $\square$

**Corollary 2.3.** *Let  $k$  be a field and  $S$  a non-trivial ring. Then any epimorphism  $k \rightarrow S$  is an isomorphism.*

**Proof.** Every non-zero vector space over a field is faithfully flat, then apply the above Corollary.  $\square$

A ring map which is both flat and an epimorphism is called a flat epimorphism.

**Corollary 2.4.** *Let  $\varphi : R \rightarrow S$  be a flat epimorphism. Then for each prime ideal  $\mathfrak{q}$  of  $S$ , the induced map  $\varphi_{\mathfrak{q}} : R_{\mathfrak{p}} \rightarrow S_{\mathfrak{q}}$  is an isomorphism where  $\mathfrak{p} = \varphi^*(\mathfrak{q})$ .*

**Proof.** By [5, Theorem 7.2],  $\varphi_{\mathfrak{q}} : R_{\mathfrak{p}} \rightarrow S_{\mathfrak{q}}$  is faithfully flat. It is also an epimorphism since the following diagram is commutative

$$\begin{array}{ccc} R & \xrightarrow{\varphi=\text{epic}} & S \\ \downarrow & & \downarrow \text{epic} \\ R_{\mathfrak{p}} & \xrightarrow{\varphi_{\mathfrak{q}}} & S_{\mathfrak{q}}. \end{array}$$

Now the assertion implies from Corollary 2.2.  $\square$

### 3. FINITE TYPE EPIMORPHISMS

Let  $\varphi : R \rightarrow S$  be a ring map. Consider the canonical ring map  $p : S \otimes_R S \rightarrow S$  which maps each pure tensor  $s \otimes s'$  into  $ss'$ . In the previous section we observed that the ideal  $\text{Ker}(p)$  is generated by the elements  $s \otimes 1 - 1 \otimes s$  where  $s \in S$ . Because if  $ss' = 0$  then we may

write  $s \otimes s' = 1 \otimes s'(s \otimes 1 - 1 \otimes s)$ . For the finite type ring maps we have the following interesting result:

**Lemma 3.1.** *If the ring map  $\varphi : R \rightarrow S$  is of finite type then  $\text{Ker}(p)$  is a finitely generated ideal.*

**Proof.** By the hypothesis there are elements  $s_1, \dots, s_n \in S$  such that  $S = R[s_1, \dots, s_n]$ . Let  $J$  be the ideal of  $S \otimes_R S$  generated by the elements  $s_i \otimes 1 - 1 \otimes s_i$  where  $1 \leq i \leq n$ . Clearly  $J \subseteq \text{Ker}(p)$ . To prove the reverse inclusion it suffices to show that for each monomial  $s_1^{d_1} \dots s_n^{d_n}$  then the element  $s_1^{d_1} \dots s_n^{d_n} \otimes 1 - 1 \otimes s_1^{d_1} \dots s_n^{d_n}$  belongs to  $J$ . To prove this we use an induction argument over  $n$ . If  $n = 1$  then we have  $s^d \otimes 1 - 1 \otimes s^d = (s \otimes 1)^d - (1 \otimes s)^d = ((s \otimes 1)^{d-1} + (s \otimes 1)^{d-2}(1 \otimes s) + \dots + (1 \otimes s)^{d-1})(s \otimes 1 - 1 \otimes s)$  which belongs to  $J$ . Let  $n > 1$ . Then we may write  $s_1^{d_1} \dots s_n^{d_n} \otimes 1 - 1 \otimes s_1^{d_1} \dots s_n^{d_n} = s_n^{d_n} \otimes 1(s_1^{d_1} \dots s_{n-1}^{d_{n-1}} \otimes 1 - 1 \otimes s_1^{d_1} \dots s_{n-1}^{d_{n-1}}) + 1 \otimes s_1^{d_1} \dots s_{n-1}^{d_{n-1}}(s_n^{d_n} \otimes 1 - 1 \otimes s_n^{d_n})$  which by the induction hypothesis and the induction step belongs to  $J$ .  $\square$

**Lemma 3.2.** *If  $\varphi : R \rightarrow S$  is an epimorphism then  $\Omega_{S/R} = 0$ .*

**Proof.** By [3, Tag 00RW],  $\Omega_{S/R} \simeq J/J^2$  where  $J$  is the kernel of the canonical ring map  $p : S \otimes_R S \rightarrow S$ . But  $J = 0$  since  $s \otimes 1 = 1 \otimes s$  for all  $s \in S$ .  $\square$

There is another characterization of epimorphisms which is so useful:

**Theorem 3.3.** *A ring map  $\varphi : R \rightarrow S$  is an epimorphism if and only if the following conditions hold.*

- (a) *The induced map  $\varphi^* : \text{Spec}(S) \rightarrow \text{Spec}(R)$  is injective.*
- (b) *For each prime ideal  $\mathfrak{q}$  of  $S$ , the induced map  $\kappa(\mathfrak{p}) \rightarrow \kappa(\mathfrak{q})$  is an epimorphism where  $\mathfrak{p} = \varphi^*(\mathfrak{q})$ .*
- (c) *The kernel of the canonical ring map  $p : S \otimes_R S \rightarrow S$  is a finitely generated ideal.*
- (d) *The module of differentials  $\Omega_{S/R}$  is zero.*

**Proof.** The implication “ $\Rightarrow$ ” is obvious since for each prime ideal  $\mathfrak{p}$  in  $R$ , the fiber  $(\varphi^*)^{-1}(\mathfrak{p})$  is homeomorphic to  $\text{Spec}(S \otimes_R \kappa(\mathfrak{p}))$ . By Corollary 2.3,  $\kappa(\mathfrak{p}) \otimes_R S$  is a field whenever it is non-trivial. Therefore,

the fiber  $(\varphi^*)^{-1}(\mathfrak{p})$  has at most one point. This implies that  $\varphi^*$  is injective. The map  $\kappa(\mathfrak{p}) \rightarrow \kappa(\mathfrak{q})$  is an epimorphism since the following diagram

$$\begin{array}{ccc} R & \xrightarrow{\varphi=\text{epic}} & S \\ \downarrow & & \downarrow \text{epic} \\ \kappa(\mathfrak{p}) & \longrightarrow & \kappa(\mathfrak{q}) \end{array}$$

is commutative. Therefore, by Corollary 2.3, it is an isomorphism. This establishes (b). The condition (c) clearly holds since  $s \otimes 1 = 1 \otimes s$  for all  $s \in S$ . In fact  $\text{Ker}(p) = 0$ . For the latter condition see Lemma 3.2. Conversely, let  $T$  be a reduced ring and let  $f, g : S \rightarrow T$  be ring maps such that  $f \circ \varphi = g \circ \varphi$ . We have  $f^* = g^*$  since  $\varphi^*$  is injective. For each prime ideal  $P$  of  $T$  let  $\mathfrak{q} = f^*(P)$  and  $\mathfrak{p} = \varphi^*(\mathfrak{q})$ . Denote by  $\varphi' : \prod_{P \in \text{Spec}(T)} \kappa(\mathfrak{p}) \rightarrow \prod_{P \in \text{Spec}(T)} \kappa(\mathfrak{q})$  the ring map induced by  $\varphi : R \rightarrow S$ . By the hypothesis (b),  $\varphi'$  is an isomorphism. Similarly, denote by  $f', g' : S' = \prod_{P \in \text{Spec}(T)} \kappa(\mathfrak{q}) \rightarrow T' = \prod_{P \in \text{Spec}(T)} \kappa(P)$  the ring maps induced by  $f$  and  $g$ , respectively. From  $f \circ \varphi = g \circ \varphi$  we have  $f' \circ \varphi' = g' \circ \varphi'$ . Therefore  $f' = g'$  since  $\varphi'$  is an isomorphism. This implies that  $\rho \circ f = \rho \circ g$  where  $\rho : T \rightarrow T'$  is the canonical ring map. But  $\rho$  is injective since  $T$  is a reduced ring. Therefore  $f = g$ . This, in particular, implies that  $\eta \circ i = \eta \circ j$  where  $\eta : S \otimes_R S \rightarrow (S \otimes_R S)_{\text{red}}$  is the canonical ring map,  $i(s) = s \otimes 1$  and  $j(s) = 1 \otimes s$ . Thus  $s \otimes 1 - 1 \otimes s$  is nilpotent for all  $s \in S$ . But the hypotheses (c) and (d) imply that  $s \otimes 1 = 1 \otimes s$  for all  $s \in S$ .  $\square$

Let  $\varphi : R \rightarrow S$  be a ring map. The induced map  $\varphi^* : \text{Spec}(S) \rightarrow \text{Spec}(R)$  is said to be *universally injective* if for any ring map  $R \rightarrow R'$  then  $\psi^* : \text{Spec}(R' \otimes_R S) \rightarrow \text{Spec}(R)$  is injective where  $\psi : R' \rightarrow R' \otimes_R S$  is the base change map. To see the definition of a formally unramified ring map please consider [3, Tag 00UN].

The following is the first main result of this article.

**Theorem 3.4.** *Let  $\varphi : R \rightarrow S$  be an of finite type ring map. Then the following conditions are equivalent.*

- (i) *The map  $\varphi$  is an epimorphism.*
- (ii) *The map  $\varphi^* : \text{Spec}(S) \rightarrow \text{Spec}(R)$  is injective and for each prime ideal  $\mathfrak{q}$  of  $S$  the base change map  $\kappa(\mathfrak{p}) \rightarrow S_{\mathfrak{q}} \otimes_R \kappa(\mathfrak{p})$  is an epimorphism where  $\mathfrak{p} = \varphi^*(\mathfrak{q})$ .*

- (iii) The map  $\varphi^*$  is universally injective and  $\Omega_{S/R} = 0$ .
- (iv) The map  $\varphi$  is formally unramified and for any field  $K$  and any ring maps  $f, g : S \rightarrow K$  if  $f \circ \varphi = g \circ \varphi$  then  $f = g$ .
- (v) The map  $\varphi^*$  is injective,  $\Omega_{S/R} = 0$  and for each prime ideal  $\mathfrak{q}$  of  $S$ , the field extension  $\kappa(\mathfrak{p}) \subseteq \kappa(\mathfrak{q})$  is purely inseparable where  $\mathfrak{p} = \varphi^*(\mathfrak{q})$ .

**Proof.** (i)  $\Rightarrow$  (ii) : By Theorem 3.3,  $\varphi^*$  is injective. Compose  $\varphi$  with the canonical ring map  $S \rightarrow S_{\mathfrak{q}}$  we then obtain the epimorphism  $R \rightarrow S_{\mathfrak{q}}$ . Epimorphisms are stable under base change. Thus the map  $\kappa(\mathfrak{p}) \rightarrow S_{\mathfrak{q}} \otimes_R \kappa(\mathfrak{p})$  is an epimorphism.

(ii)  $\Rightarrow$  (i) : Let  $\mathfrak{q}$  be a prime ideal of  $S$  and let  $\mathfrak{p} = \varphi^*(\mathfrak{q})$ . By [11, Lemma 2.1],  $S_{\mathfrak{q}} \otimes_R \kappa(\mathfrak{p}) \neq 0$  and so by Corollary 2.3,  $\kappa(\mathfrak{p}) \simeq S_{\mathfrak{q}} \otimes_R \kappa(\mathfrak{p}) \simeq S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}}$ . Thus  $S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}}$  is a field and so  $\mathfrak{p}S_{\mathfrak{q}} = \mathfrak{q}S_{\mathfrak{q}}$ . Hence the induced map  $\kappa(\mathfrak{p}) \rightarrow \kappa(\mathfrak{q})$  is an isomorphism. Moreover, by [3, Tag 00RV], we have  $0 = \Omega_{S_{\mathfrak{q}} \otimes_R \kappa(\mathfrak{p})/\kappa(\mathfrak{p})} \simeq \Omega_{S_{\mathfrak{q}}/R} \otimes_R \kappa(\mathfrak{p})$ . This implies that  $(\mathfrak{q}S_{\mathfrak{q}})\Omega_{S_{\mathfrak{q}}/R} = \Omega_{S_{\mathfrak{q}}/R}$ . By [3, Tags 00RT, 00RZ],  $\Omega_{S_{\mathfrak{q}}/R}$  as  $S_{\mathfrak{q}}$ -module is finitely generated and so by the Nakayama lemma,  $\Omega_{S_{\mathfrak{q}}/R} = 0$ . This means that  $\Omega_{S/R} = 0$  since  $(\Omega_{S/R})_{\mathfrak{q}} \simeq \Omega_{S_{\mathfrak{q}}/R} = 0$ . Now using Lemma 3.1 and Theorem 3.3 then we conclude that  $\varphi$  is an epimorphism.

(i)  $\Rightarrow$  (iii) : Epimorphisms are stable under base change, then apply Theorem 3.3.

(iii)  $\Rightarrow$  (iv) : The ring map  $\varphi$  is formally unramified if and only if  $\Omega_{S/R} = 0$ , see [3, Tag 00UO]. Now consider the induced ring maps  $f', g' : K \otimes_R S \rightarrow K$  which map each pure tensor  $a \otimes s$  into  $af(s)$  and  $ag(s)$ , respectively. The map  $\psi^*$  is injective since  $\varphi^*$  is universally injective where  $\psi : K \rightarrow K \otimes_R S$  is the base change of  $(\varphi, h)$  with  $h = f \circ \varphi : R \rightarrow K$ . Moreover, by [11, Lemma 2.1],  $K \otimes_R S$  is a non-trivial ring. Therefore the prime spectrum of  $K \otimes_R S$  is a single-point set. This, in particular, implies that  $(f')^{-1}(0) = (g')^{-1}(0)$ . But for each  $s \in S$ ,  $f'(f(s) \otimes 1 - 1 \otimes s) = 0$ . Thus  $g'(f(s) \otimes 1 - 1 \otimes s) = 0$ . Therefore  $f(s) = g(s)$  for all  $s \in S$ .

(iv)  $\Rightarrow$  (i) : Let  $\mathfrak{p}$  be a prime ideal of  $S \otimes_R S$ . From the hypotheses we obtain  $\pi_{\mathfrak{p}} \circ i = \pi_{\mathfrak{p}} \circ j$  where  $\pi_{\mathfrak{p}} : S \otimes_R S \rightarrow \kappa(\mathfrak{p})$  and  $i, j : S \rightarrow S \otimes_R S$  are the canonical ring maps. Therefore for each  $s \in S$ ,  $s \otimes 1 - 1 \otimes s$  is a nilpotent element. Thus, by Lemma 3.1,  $J$  is a nilpotent ideal where  $J$  is the kernel of the canonical ring map  $S \otimes_R S \rightarrow S$ . But  $\Omega_{S/R} \simeq J/J^2$ . Thus  $J = J^2$  and so  $J = 0$ . Therefore  $s \otimes 1 = 1 \otimes s$  for all  $s \in S$ . Hence, by Theorem 2.1,  $\varphi$  is an epimorphism.

(i)  $\Rightarrow$  (v) : See Theorem 3.3.

(v)  $\Rightarrow$  (iv) : We have  $f^{-1}(0) = g^{-1}(0)$  since  $f \circ \varphi = g \circ \varphi$  and  $\varphi^*$  is injective. Let  $\mathfrak{q} = f^{-1}(0)$ . For each  $s \in S$ , by the hypotheses, there is



a natural number  $n \geq 0$  such that  $(s/1 + \mathfrak{q}S_{\mathfrak{q}})^{p^n}$  is in the image of the induced map  $\kappa(\mathfrak{p}) \rightarrow \kappa(\mathfrak{q})$  where  $\mathfrak{p} = \varphi^*(\mathfrak{q})$  and  $p$  is the characteristic of  $\kappa(\mathfrak{p})$ . Therefore there are elements  $r \in R$  and  $t \in R \setminus \mathfrak{p}$  such that  $\varphi(t)s^{p^n} - \varphi(r) \in \mathfrak{q}$ . Thus  $f(s)^{p^n} = (f \circ \varphi)(r)(f \circ \varphi)(t)^{-1} = g(s)^{p^n}$ . This implies that  $(f(s) - g(s))^{p^n} = 0$  since the characteristic of  $K$  is equal to  $p$ . Therefore  $f(s) = g(s)$  for all  $s \in S$ .  $\square$

#### 4. NEW RESULTS ON FLAT EPIMORPHISMS

For any two ring maps  $\varphi : R \rightarrow A$  and  $\psi : R \rightarrow B$  we say that  $\varphi \sim \psi$  if there exists an isomorphism of rings  $\theta : A \rightarrow B$  such that  $\theta \circ \varphi = \psi$ . Clearly it is an equivalence relation. Each equivalence class is called an isomorphism class. For a given ring map  $\varphi : R \rightarrow A$  we shall denote by  $\mathcal{F}_{\varphi}$  the set of ideals of  $R$  whose extensions under  $\varphi$  are equal to  $A$ . The family  $\mathcal{F}_{\varphi}$  is an idempotent topologizing system on the ring  $R$ , see [10, Theorem 3.2].

**Theorem 4.1.** *The collection of isomorphism classes of flat epimorphisms with a fixed source is a set.*

**Proof.** Let  $R$  be a ring and consider  $[\varphi] \rightsquigarrow \mathcal{F}_{\varphi}$  which is a well-defined and injective map from the collection of isomorphism classes of flat epimorphisms with source  $R$  into the set of idempotent topologizing systems on the ring  $R$ . Because for any two flat epimorphisms  $\varphi : R \rightarrow A$  and  $\psi : R \rightarrow B$  we have  $[\varphi] = [\psi]$  if and only if  $\mathcal{F}_{\varphi} = \mathcal{F}_{\psi}$ . Indeed, the implication “ $\Rightarrow$ ” holds more generally (not just for the flat epimorphisms). The converse implies from [10, Corollary 3.3].  $\square$

**Lemma 4.2.** *Let  $f : R \rightarrow T$  be an injective ring map which factors as  $R \xrightarrow{g} S \xrightarrow{h} T$  where  $g$  is a flat epimorphism. Then  $h$  is injective.*

**Proof.** We have the factorization  $S \xrightarrow{\simeq} S \otimes_R R \xrightarrow{1 \otimes f} S \otimes_R T \xrightarrow{\simeq} S \otimes_R (S \otimes_S T) \xrightarrow{\simeq} (S \otimes_R S) \otimes_S T \xrightarrow{p \otimes 1} S \otimes_S T \xrightarrow{\simeq} T$  for the map  $h$  where the unnamed arrows are the canonical isomorphisms. By Theorem 2.1, the canonical ring map  $p : S \otimes_R S \rightarrow S$  is an isomorphism. The map  $1 \otimes f : S \otimes_R R \rightarrow S \otimes_R T$  is also injective since  $S$  is  $R$ -flat. Therefore  $h$  is injective.  $\square$

**Theorem 4.3.** *Let  $R$  be a ring. Then there exist a ring  $M(R)$  and a canonical injective flat epimorphism  $\eta : R \rightarrow M(R)$  such that every  $R$ -algebra whose structure morphism is an injective flat epimorphism can be canonically embedded in  $M(R)$ .*

**Proof.** Consider the collection of isomorphism classes of injective flat epimorphisms with source  $R$ . Note that, by Theorem 4.1, it is actually a set. Therefore, by the axiom of choice, we may choose a set  $\{\varphi_\alpha : R \rightarrow A_\alpha\}_{\alpha \in I}$  whose elements have been precisely picked up from the distinct isomorphism classes. We say that  $\alpha \leq \beta$  if there exists a ring map  $\lambda_{\alpha,\beta} : A_\alpha \rightarrow A_\beta$  such that  $\varphi_\beta = \lambda_{\alpha,\beta} \circ \varphi_\alpha$ . Note that such a map  $\lambda_{\alpha,\beta}$ , if it exists, is unique since  $\varphi_\alpha$  is an epimorphism. The poset  $(I, \leq)$  is also directed. Because for any pair  $(\alpha, \beta)$  of elements of  $I$  consider the following pushout diagram

$$\begin{array}{ccc} R & \xrightarrow{\varphi_\alpha} & A_\alpha \\ \downarrow \varphi_\beta & & \downarrow \\ A_\beta & \longrightarrow & A_\alpha \otimes_R A_\beta \end{array}$$

clearly  $R \rightarrow A_\alpha \otimes_R A_\beta$  is a flat epimorphism since flat morphisms and epimorphisms are stable under base change and composition. It is also injective since each  $A_\alpha$  is  $R$ -flat. Thus there exists an element  $\gamma \in I$  such that  $\varphi_\gamma$  belonging to the isomorphism class  $[R \rightarrow A_\alpha \otimes_R A_\beta]$  and so  $\alpha, \beta \leq \gamma$ . Therefore  $(A_\alpha, \lambda_{\alpha,\beta})$  is an inductive (direct) system of  $R$ -algebras over the directed poset  $(I, \leq)$ . The canonical map  $\eta : R \rightarrow M(R)$  is an injective flat epimorphism where  $M(R) = \operatorname{colim}_{\alpha \in I} A_\alpha$  is the inductive limit (colimit) of the system  $(A_\alpha, \lambda_{\alpha,\beta})$ . If  $\varphi : R \rightarrow A$  is an injective flat epimorphism then there is a (unique) ring map  $h : A \rightarrow M(R)$  such that  $h \circ \varphi = \eta$ . By Lemma 4.2,  $h$  is injective.  $\square$

The ring  $M(R)$ , obtained in Theorem 4.3, is called the *Lazard* ring of  $R$ .

In what follows we shall consider the canonical ring maps  $\eta : R \rightarrow M(R)$  and  $R \rightarrow T(R)$  where  $\eta$  as in Theorem 4.3 and  $T(R)$  is the total ring of fractions of  $R$ . Note that the canonical map  $R \rightarrow T(R)$  is an injective flat epimorphism and so, by Theorem 4.3,  $T(R)$  can be canonically embedded in  $M(R)$ . Under some circumstances on the ring  $R$  and also by using the Gabriel localization theory then we conclude

that the canonical embedding  $T(R) \hookrightarrow M(R)$  is bijective:

**Theorem 4.4.** *Let  $R$  be a ring with the property that every f.g. and faithful ideal of it has a regular element. Then there is a unique isomorphism of rings  $\psi : M(R) \rightarrow T(R)$  such that  $\pi = \psi \circ \eta$ .*

**Proof.** We claim that  $\mathcal{F}_\pi = \mathcal{F}_\eta$ . By Theorem 4.3, the map  $\eta$  factors through  $T(R)$  and so  $\mathcal{F}_\pi \subseteq \mathcal{F}_\eta$ . Conversely, let  $I \in \mathcal{F}_\eta$ . Then we may write  $1 = \sum_{i=1}^n s_i \eta(a_i)$  where  $s_i \in M(R)$  and  $a_i \in I$  for all  $i$ . Clearly  $\langle a_1, \dots, a_n \rangle$  is a faithful ideal of  $R$  since  $\eta$  is injective. Therefore, by the hypothesis,  $I$  meets  $R \setminus Z(R)$ . This means that  $IT(R) = T(R)$ . Therefore  $\mathcal{F}_\eta = \mathcal{F}_\pi$ . By [10, Corollary 3.3], there is an isomorphism of rings  $\psi : M(R) \rightarrow T(R)$  such that  $\pi = \psi \circ \eta$ . Such  $\psi$  is unique since  $\eta$  is an epimorphism.  $\square$

The following is a well-known result.

**Lemma 4.5.** *Let  $R$  be a ring. If  $f = \sum_{i=0}^n a_i x^i$  is a zero-divisor element of the polynomial ring  $R[x]$  then there is a non-zero element  $c \in R$  such that  $cf = 0$ .*

**Proof.** There is a non-zero element  $g = \sum_{j=0}^s b_j x^j \in R[x]$  such that  $gf = 0$ . If  $\deg(g) = 0$  then take  $c = b_0$ . Suppose  $\deg(g) = s > 0$ . We may assume that  $b_s a_k \neq 0$  for some  $k$  because if  $b_s$  vanishes all of the coefficients of  $f$  then there is nothing to prove. Therefore  $ga_k \neq 0$ . Let  $t$  be the largest index such that  $ga_t \neq 0$ . Therefore  $g \sum_{i=0}^t a_i x^i = 0$ . It follows that  $b_s a_t = 0$ . Thus  $\deg(ga_t) < s$  and clearly  $(ga_t)f = 0$ . Now the assertion implies from the induction hypothesis.  $\square$

The hypotheses of Theorem 4.4 are not limitative at all:

**Theorem 4.6.** *Let  $R$  be a ring. Then  $T(R[x])$  is canonically isomorphic to  $M(R[x])$ .*

**Proof.** Let  $I = \langle f_1, \dots, f_n \rangle$  be a f.g. and faithful ideal of  $A = R[x]$  where  $\deg(f_i) = d_i$  with  $d_1 \leq d_2 \leq \dots \leq d_n$ . By Theorem 4.4, it suffices

to show that  $I$  meets  $A \setminus Z(A)$ . Take  $f'_1 = f_1$  and for each  $i \geq 2$  take  $f'_i = x^{s_i} f_i$  where  $s_i = d_1 + \dots + d_{i-1} + (i-1)$ . Then consider the polynomial  $g = \sum_{i=1}^n f'_i$  which belongs to  $I$ . We have  $g \in A \setminus Z(A)$ . If not, then by Lemma 4.5, we may find a non-zero element  $c \in R$  such that  $cg = 0$ . It follows that  $c \in \bigcap_{i=1}^n \text{Ann}_A(f_i) = 0$ , a contradiction.  $\square$

**Corollary 4.7.** *Let  $R$  be a ring and let  $A = R[x]$ . Then every  $A$ -algebra whose structure morphism is an injective flat epimorphism can be canonically embedded in  $T(A)$ .*

**Proof.** It is an immediate consequence of Theorems 4.3 and 4.6.  $\square$

We need the following Lemmata, which are interesting in their own right, to prove Theorem 4.11.

**Lemma 4.8.** *Let  $R$  be a ring and consider the  $R$ -algebra homomorphism  $\psi : R[x_1, \dots, x_n] \rightarrow R$  which maps each  $x_i$  into  $c_i$  where  $c_i \in R$  for all  $i$ . Then  $\text{Ker}(\psi) = \langle x_i - c_i : 1 \leq i \leq n \rangle$ .*

**Proof.** We use an induction argument on  $n$ . If  $n = 1$  then the assertion implies from the division algorithm. Let  $n > 1$ . The map  $\psi$  factors as

$$R[x_1, \dots, x_n] \xrightarrow{\varphi} R[x_1, \dots, x_{n-1}] \xrightarrow{\lambda} R$$

where the  $R$ -algebra maps  $\varphi$  and  $\lambda$  are defined for each  $1 \leq i \leq n-1$  as  $\varphi(x_i) = x_i$ ,  $\lambda(x_i) = c_i$  and  $\varphi(x_n) = c_n$ . We have  $\text{Ker}(\psi) = \psi^{-1}(0) = \varphi^{-1}(\lambda^{-1}(0)) = \varphi^{-1}(\text{Ker}(\lambda))$ . By the induction hypothesis,  $\text{Ker}(\lambda) = \langle x_i - c_i : 1 \leq i \leq n-1 \rangle$ . Take  $f(x_1, \dots, x_n) \in \text{Ker}(\psi)$ .

Write  $f(x_1, \dots, x_n) = \sum_{j=0}^d f_j(x_1, \dots, x_{n-1})x_n^j = \sum_{j=0}^d f_j(x_1, \dots, x_{n-1})c_n^j + \sum_{j=1}^d f_j(x_1, \dots, x_{n-1})(x_n^j - c_n^j) = \varphi(f(x_1, \dots, x_n)) + \left( \sum_{j=1}^d f_j(x_1, \dots, x_{n-1})(x_n^{j-1} + x_n^{j-2}c_n + \dots + c_n^{j-1}) \right)(x_n - c_n) = \sum_{i=1}^{n-1} h_i(x_1, \dots, x_{n-1})(x_i - c_i) + h(x_1, \dots, x_n)(x_n - c_n)$  which belongs to  $\langle x_i - c_i : 1 \leq i \leq n \rangle$ .  $\square$

**Lemma 4.9.** *Let  $R$  be a ring and let  $M$  be a flat  $R$ -module. Then  $(I : J)M = IM : J$  for every ideals  $I$  and  $J$  of  $R$  with  $J$  finitely generated.*

**Proof.** By [3, Tag 0BBY], it suffices to show that  $(I : a)M = IM : a$  for all  $a \in R$ . Clearly  $(I : a)M \subseteq IM : a$ . To see the reverse inclusion take  $m \in IM : a$  then the element  $(1 + I) \otimes m$  is in the kernel of the endomorphism  $\psi : R/I \otimes_R M \rightarrow R/I \otimes_R M$  which maps each pure tensor  $(r + I) \otimes x$  into  $(ar + I) \otimes x$ . By applying the exact functor  $- \otimes_R M$  to the exact sequence  $(I : a)/I \longrightarrow R/I \xrightarrow{a} R/I$  we obtain the follow-

ing exact sequence  $(I : a)/I \otimes_R M \longrightarrow R/I \otimes_R M \xrightarrow{\psi} R/I \otimes_R M$ . It follows that  $(1 + I) \otimes m = \sum_i (b_i + I) \otimes m_i$  where  $b_i \in I : a$  and  $m_i \in M$ . Now consider their image under the canonical isomorphism  $R/I \otimes_R M \rightarrow M/IM$  then we will have  $m - \sum_i b_i m_i \in IM \subseteq (I : a)M$  and so  $m \in (I : a)M$ .  $\square$

It is worthy to mention that the converse of Lemma 4.9 also holds. More precisely, if  $(I : a)M = IM : a$  for each  $a \in R$  and for every ideal  $I$  of  $R$  then  $M$  is  $R$ -flat.

**Lemma 4.10.** *Let  $I$  be an ideal of a ring  $R$  such that  $I[x] = \langle f_1, \dots, f_n \rangle$  is a f.g. ideal of  $R[x]$ . Then  $I$  is a f.g. ideal.*

**Proof.** We have  $I = \langle c_1, \dots, c_n \rangle$  where  $c_i$  is the constant term of the polynomial  $f_i$  for all  $i$ .  $\square$

**Theorem 4.11.** *Every injective flat epimorphism of rings which is also of finite type then it is of finite presentation.*

**Proof.** Let  $R$  be a ring and let  $I$  be an ideal of the polynomial ring  $B = R[x_1, \dots, x_n]$  such that the canonical map  $\pi' : R \rightarrow B/I$  is an injective flat epimorphism. We shall prove that  $I$  is a f.g. ideal. Consider the following push-out diagram

$$\begin{array}{ccc} R & \xrightarrow{\pi'} & B/I \\ \downarrow & & \downarrow \\ A & \longrightarrow & B/I \otimes_R A \end{array}$$

where  $A = R[x_{n+1}]$ . The map  $A \rightarrow B/I \otimes_R A$  is a flat epimorphism since flat maps and epimorphisms are stable under base change. It is also injective since  $A$  is  $R$ -flat. But  $B/I \otimes_R A$  is canonically isomorphic to  $B'$  and the canonical map  $\pi : A \rightarrow B'$  factors as  $A \rightarrow B/I \otimes_R A \simeq B'$  where  $B' = B[x_{n+1}]/I^e$  and  $I^e = I[x_{n+1}]$ . Therefore  $\pi$  is an injective flat epimorphism and so, by Corollary 4.7, there is an injective ring map  $\psi : B' \rightarrow T(A)$  such that  $\psi \circ \pi : A \rightarrow T(A)$  is the canonical map. We show that  $I^e$  is a f.g. ideal. For each  $1 \leq i \leq n$ , we may write  $\psi(x_i + I^e) = f_i/g$  where  $f_i \in A$  and  $g \in T = A \setminus Z(A)$ . Let  $J$  be the ideal of  $A$  generated by all of the  $f_i$ . By Lemma 4.9,  $(Ag : J)B' = (B'g : J)$ . But  $(B'g : J) = B'$  because  $\psi(f_i - gx_i + I^e) = 0$ . Hence, there are elements  $h \in I^e$  and  $h' \in (Ag : J)A[x_1, \dots, x_n]$  such that  $1 = h + h'$ .

Write  $h' = \sum_{j=1}^m s_j h_j$  where  $s_j \in (Ag : J)$  and  $h_j \in A[x_1, \dots, x_n]$ . It follows that  $f_i s_j = h_{ij} g$  for some elements  $h_{ij} \in A$ . Then  $x_i s_j - h_{ij} \in I^e$  since  $\psi(x_i s_j - h_{ij} + I^e) = 0$ . Let  $L$  be the ideal of  $A[x_1, \dots, x_n]$  generated by  $h$  and all of the  $x_i s_j - h_{ij}$ . We claim that  $I^e = L$ . To prove the claim it suffices to show that for each prime ideal  $\mathfrak{p}$  of  $A[x_1, \dots, x_n]$  then  $(I^e)_{\mathfrak{p}} = L_{\mathfrak{p}}$ . It holds if  $L$  is not contained in  $\mathfrak{p}$  since  $L \subseteq I^e$ . Suppose  $L \subseteq \mathfrak{p}$ . Then  $h' \notin \mathfrak{p}$  hence there is some  $k$  such that  $s_k \notin \mathfrak{p}$ . Let  $\mathfrak{q} = A \cap \mathfrak{p}$  and then consider the  $A_{\mathfrak{q}}$ -algebra homomorphism  $\varphi : A_{\mathfrak{q}}[x_1, \dots, x_n] \rightarrow A_{\mathfrak{q}}$  which maps each  $x_i$  into  $h_{ik}/s_k$ . By Lemma 4.8,  $\text{Ker}(\varphi) = \langle x_i - h_{ik}/s_k : 1 \leq i \leq n \rangle = \langle s_k x_i - h_{ik} : 1 \leq i \leq n \rangle \subseteq L_{\mathfrak{q}}$  where  $L_{\mathfrak{q}}$  is the extension of  $L$  under the canonical ring map  $A[x_1, \dots, x_n] \rightarrow A_{\mathfrak{q}}[x_1, \dots, x_n] \simeq (A[x_1, \dots, x_n])_{\mathfrak{q}}$ . By applying  $s_k f_i = h_{ik} g$  with  $1 \leq i \leq n$  then we may factor the map  $\pi'' \circ \varphi$  as

$$A_{\mathfrak{q}}[x_1, \dots, x_n] \longrightarrow A_{\mathfrak{q}}[x_1, \dots, x_n]/(I^e)_{\mathfrak{q}} \xrightarrow{\cong} B'_{\mathfrak{q}} \xrightarrow{\psi_{\mathfrak{q}}} (T(A))_{\mathfrak{q}} \xrightarrow{\cong} S^{-1}(A_{\mathfrak{q}})$$

where  $S$  is the image of  $A \setminus Z(A)$  under the canonical map  $A \rightarrow A_{\mathfrak{q}}$  and  $\pi''$  is the canonical map  $A_{\mathfrak{q}} \rightarrow S^{-1}(A_{\mathfrak{q}})$  which is injective since  $S \subseteq A_{\mathfrak{q}} \setminus Z(A_{\mathfrak{q}})$ . Therefore  $(I^e)_{\mathfrak{q}} = \text{Ker}(\varphi)$ . This implies that  $(I^e)_{\mathfrak{q}} = L_{\mathfrak{q}}$ . But  $(I^e)_{\mathfrak{p}}$  (resp.  $L_{\mathfrak{p}}$ ) is the extension of  $(I^e)_{\mathfrak{q}}$  (resp.  $L_{\mathfrak{q}}$ ) under the canonical map  $(A[x_1, \dots, x_n])_{\mathfrak{q}} \rightarrow (A[x_1, \dots, x_n])_{\mathfrak{p}}$ . Thus  $(I^e)_{\mathfrak{p}} = L_{\mathfrak{p}}$ . This establishes the claim. Therefore  $I^e$  and so  $I$ , by Lemma 4.10, are f.g. ideals.  $\square$

The following is an immediate consequence of Theorem 4.11.

**Corollary 4.12.** *Let  $S$  be a multiplicative subset of a ring  $R$  such that the canonical map  $\pi : R \rightarrow S^{-1}R$  is injective. Then  $\pi$  is of finite presentation whenever it is of finite type.  $\square$*

In the following result the functorial aspects of Lazard rings are explored.

**Lemma 4.13.** *Let  $\varphi : R \rightarrow S$  be a flat ring map. Then there is a unique flat ring map  $M(\varphi) : M(R) \rightarrow M(S)$  such that  $M(\varphi) \circ \eta_R = \eta_S \circ \varphi$ . The map  $M(\varphi)$  is injective whenever  $\varphi$  is so. If  $\varphi$  is an injective flat epimorphism then  $M(\varphi)$  is bijective.*

**Proof.** The uniqueness of such a map  $M(\varphi)$  is obvious since  $\eta_R$  is an epimorphism. To show its existence we shall consider the following push-out diagram

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & S \\ \downarrow \eta_R & & \downarrow \lambda \\ M(R) & \xrightarrow{\lambda'} & M(R) \otimes_R S. \end{array}$$

The canonical ring map  $\lambda$  is a flat epimorphism since flat maps and epimorphisms are stable under base change. It is also injective since  $\eta_R$  is injective and  $S$  is  $R$ -flat. Therefore, by Theorem 4.3, there is a ring map  $h : M(R) \otimes_R S \rightarrow M(S)$  such that  $\eta_S = h \circ \lambda$ . Take  $M(\varphi) = h \circ \lambda'$  which is the desired map. By Lemma 4.2,  $h$  is injective and so by [12, Lemma 4.2], it is flat. The map  $\lambda'$  is also flat since flat maps are stable under base change. Therefore  $M(\varphi)$  is flat. If  $\varphi$  is injective then the map  $\lambda'$  is also injective since  $M(R)$  is  $R$ -flat. Therefore  $M(\varphi)$  is injective too. Finally, if  $\varphi$  is an injective flat epimorphism then it is also true for the map  $\eta_S \circ \varphi : R \rightarrow M(S)$ . Therefore, by Theorem 4.3, there is a ring map  $\psi : M(S) \rightarrow M(R)$  such that  $\eta_R = \psi \circ (\eta_S \circ \varphi)$ . Clearly  $\psi \circ M(\varphi)$  and  $M(\varphi) \circ \psi$  are the identity maps.  $\square$

The following result strongly bounds the size of the Lazard rings:

**Corollary 4.14.** *Let  $R$  be a ring. Then the ring  $M(R)$  can be canonically embedded in  $T(R[x])$ .*

**Proof.** By Lemma 4.13, the map  $M(\epsilon)$  is injective where  $\epsilon : R \rightarrow R[x]$  is the canonical map which is injective and flat. By Theorem 4.6,

$M(R[x])$  is canonically isomorphic to  $T(R[x])$ . Then the composition

$$M(R) \xrightarrow{M(\epsilon)} M(R[x]) \xrightarrow{\simeq} T(R[x])$$

is the desired embedding.  $\square$

**Corollary 4.15.** *Let  $\varphi : R \rightarrow S$  be an injective flat epimorphism. Then there is a unique injective ring map  $\psi : S \rightarrow T(R[x])$  such that the following diagram is commutative*

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & S \\ \downarrow \epsilon & & \downarrow \psi \\ R[x] & \xrightarrow{\pi} & T(R[x]) \end{array}$$

where  $\epsilon$  and  $\pi$  are the canonical maps.

**Proof.** It is an immediate consequence of the above results.  $\square$

In summary, for a given ring  $R$ , we have up to isomorphisms the following chain of subrings  $R \subseteq T(R) \subseteq M(R) \subseteq T(R[x]) = M(R[x])$ .

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